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SECTION 1. Theoretical research in mathematics.

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AN INTEGRAL EQUATION WITH A SPECIAL KERNEL

***Abstract**: In this paper we investigate the question of the spectrum and the solvability of equation Volterra under conditions λ∈C and t∈R_+.*

***Key words**: range, solvability, Volterra integral equation.*
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ОБ ОДНОМ ИНТЕГРАЛЬНОМ УРАВНЕНИИ С ОСОБЫМ ЯДРОМ

***Аннотация**: В данной работе исследуется вопрос спектра и разрешимость уравнения Вольтерра при условиях λ∈C и t∈R_+.*

***Ключевые слова**: спектр, разрешимость, уравнение Вольтерра.*

Statement of the problem. Consider the integral equation of Volterra type of the second kind:

$$\mu(t) - \lambda \int_0^t \frac{\mu(\tau)d\tau}{\tau^{\alpha}(t-\tau)^{1-\alpha}} = f(t), t \in \mathbb{R}_+, \quad 0 < \alpha < 1, \lambda \in \mathbb{C}. \qquad (1)$$

We assume that the right-hand side and the solution of equation (1) belong to the class of integrable functions with corresponding weights:

$$e^{-t}f(t) \in L_1(\mathbb{R}_+), \quad t^{-\alpha}e^{-t}\mu(t) \in L_1(\mathbb{R}_+). \qquad (2)$$

Objective. To investigate the solvability of special Volterra integral equation of the second kind (1) under the conditions (2).

We write the equation (1) in the form

$$\mu(t) - \lambda \int_0^t \mathcal{K}(t,\tau)\mu(\tau)d\tau = f(t), \quad t \in \mathbb{R}_+, \qquad (3)$$

where

$$\mathcal{K}(t,\tau) = \frac{1}{\tau^{\alpha}(t-\tau)^{1-\alpha}}, 0 < \tau < t < \infty. \qquad (4)$$

Note that the norm of the integral operator defined by the kernel $\mathcal{K}(t,\tau)$and acting in the space of integrable functions is $\frac{\pi}{sin\pi\alpha} \neq 0$. The validity of this follows from

$$\int_0^t \frac{d\tau}{\tau^{\alpha}(t-\tau)^{1-\alpha}} = B(\alpha, 1-\alpha) = \frac{\pi}{\sin \pi\alpha} > 0.$$

If to enter function k (z) by a formula

$$k(z)=\begin{cases}0, & 0<z<1;\\ \dfrac{1}{(z-1)^{1-\alpha}}, & 1<z<\infty,\end{cases} \qquad (5)$$

we can rewrite equation (3) as

$$\mu(t)-\lambda\int_0^t k\left(\frac{t}{\tau}\right)\mu(\tau)\frac{d\tau}{\tau}=f(t). \qquad (6)$$

We investigate the homogeneous integral equation and corresponds to (6):

$$\mu(t)-\lambda\int_0^t k\left(\frac{t}{\tau}\right)\mu(\tau)\frac{d\tau}{\tau}=0. \qquad (7)$$

Applying the Mellin transform [7], taking into account the convolution theorem, we obtain

$$\tilde{\mu}(s)[1-\lambda\tilde{k}(s)]=0, s=s_1+is_2,$$

where

$$\tilde{\mu}(s)=\int_0^\infty \mu(\tau)\tau^{s-1}d\tau, Re\, s>0,$$

image of function $\mu(t)$ and of the kernel has a form

$$\tilde{k}(s)=\int_0^1 z^{-s-\alpha}(1-z)^{\alpha-1}dz=B(\alpha,-s+1-\alpha),$$

$$Re\, s<1-\alpha, \qquad (8)$$

here $B\,(x,y)$ −Veta function.

The presence and form of the eigenfunctions of the homogeneous integral equation (7) is determined by the presence and quantity of roots of the equation transcendence

$$1-\lambda\tilde{k}(s)=0 \qquad (9)$$

concerning the complex parameter s.

Investigate in more detail the question of the roots of equation (9), i.e., according to (8) of the equation

$$\lambda\cdot B(\alpha,-s+1-\alpha)=1. \qquad (10)$$

Given the properties of the beta function $B\,(\alpha,\beta)=B\,(\beta,\alpha)$ and using its representation as a series [9], we obtain

$$B(1-\alpha-s,\alpha)$$
$$=\frac{1}{\alpha}\sum_{n=0}^\infty(-1)^n\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)\ldots(\alpha-n)}{n!\,(n+1-\alpha-s)}$$
$$=$$

$$=\frac{1}{1-\alpha-s}+\sum_{n=1}^\infty\frac{b_n}{n+1-\alpha-s}, \qquad (11)$$

where

$$b_n=\frac{(1-\alpha)(2-\alpha)\ldots(n-\alpha)}{n!}=\prod_{k=1}^n\left(1-\frac{\alpha}{k}\right)>0.$$

Thus, the image of the kernel of the integral equation (7) can be represented as follows

$$B(-s+1-\alpha,\alpha)=\sum_{n=0}^\infty\frac{b_n}{n+1-\alpha-s}, \quad b_0=1.$$

Writing the equation (11) in the form

$$\frac{1}{\lambda_1+i\lambda_2}=\sum_{n=0}^\infty\frac{b_n}{n+1-\alpha-s_1-is_2},$$

obtain the relations

$$\frac{\lambda_1}{|\lambda|^2}=\sum_{n=0}^\infty b_n\frac{n+(1-\alpha-s_1)}{(n+1-\alpha-s_1)^2+s_2^2},$$

$$-\frac{\lambda_2}{|\lambda|^2}$$
$$=s_2\sum_{n=0}^\infty b_n\frac{n+(1-\alpha-s_1)}{(n+1-\alpha-s_1)^2+s_2^2}. \qquad (12)$$

We have the following

Proposition 1. For all values of s, such that $Re\, s=s_1<1-\alpha$, values sum on the right equalities (12) are positive. This means that the value of $\lambda_1=Re\lambda>0$ and $\lambda_2=Im\lambda$ has a sign equal to antipositive sign of number $s_2=Ims$.

Justice of the proposition 1 at once follows from representation of Beta function (11), ratios (12) and conditions that to $b_n>00$ for $\forall\, n=0,1,2,\ldots$

Theorem 1. $\forall\,\lambda$ at $Re\,\lambda\geq\frac{\pi}{\sin\pi\alpha}$ homogeneous integral equation (7) has a nontrivial solution of the form $\mu(t)=t^{-s^*}$, where s^* is defined as the root of the equation (9) and $Re\, s^*<1-\alpha$. If $Re\,\lambda<\frac{\pi}{\sin\pi\alpha}$, then the homogeneous equation (7) has only the trivial solution.

Consider the homogeneous equation (6):

$$\mu(t)-\lambda\int_0^t k\left(\frac{t}{\tau}\right)\mu(\tau)\frac{d\tau}{\tau}=f(t).$$

Applying to both sides of this equation Mellin transform, we obtain

$$\tilde{\mu}(s)[1-\lambda\tilde{k}(s)]=\tilde{f}(s),$$

where $\tilde{k}(s)$ is determined from the equation (8), and $\tilde{f}(s)$ Mellin transform of the function $f\,(t)$:

$$\tilde{f}(s) = \int_0^\infty f(t)t^{s-1}dt\,, \qquad Re\ s > \gamma,$$

and the parameter γ is chosen so that

$$\int_0^\infty |f(t)|t^{\gamma-1}dt < \infty.$$

Thus, a particular solution of equation (6) has the form

$$\mu(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\tilde{f}(s)}{1-\lambda\tilde{k}(s)}t^{-s}ds\,, \quad \gamma < Re\ s < 1-\alpha, \qquad (13)$$

here $\gamma < \sigma < 1-\alpha$ is chosen so that $1-\lambda\tilde{k}(s) \neq 0$for a given value λ and the integral is taken along the line $Re\ s = \sigma$, parallel to the imaginary axis of the s −plane and is understood in the sense of principal value.

Transform a particular solution (13). To do this, we use the relation

$$\frac{\tilde{f}(s)}{1-\lambda\tilde{k}(s)} = \tilde{f}(s) + \frac{\lambda\tilde{k}(s)}{1-\lambda\tilde{k}(s)}\tilde{f}(s).$$

If we now introduce the notation

$$\tilde{r}(s) = \frac{\tilde{k}(s)}{1-\lambda\tilde{k}(s)},$$

then, using the convolution formula for the Mellin transform [1], we obtain

$$\mu(t) = f(t) + \lambda\int_0^\infty r\left(\frac{t}{\tau}\right)f(\tau)\frac{d\tau}{\tau}, \qquad (14)$$

where

$$r(\theta) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\tilde{k}(s)}{1-\lambda\tilde{k}(s)}\theta^{-s}ds\,, \quad \gamma < Re\ s < 1-\alpha. \quad (15)$$

At $0 < \theta < 1$ in the contour of integration include a semi-circle, lying in the left half. In this case, if $Re\,\lambda > 0$, the integrand has a unique singularity at $-s^*$, which is a zero of $1-\lambda\tilde{k}(s)$ and a simple pole of the function $\tilde{r}(s)$.

Thus, in the case of $Re\,\lambda > 0$ we have

$$r(\theta) = l(-s^*)\theta^{-s^*}, \qquad 0 < \theta < 1, \qquad (16)$$

where $l(-s^*)-$ the inverse of the logarithmic derivative of the function $\tilde{k}(s)$ at $s = -s^*$:

$$l(-s^*) = -\frac{\tilde{k}(-s^*)}{\tilde{k}'(-s^*)}. \qquad (17)$$

Hence, from (14) that a partial solution inhomogeneous integral equation (1) can be written as

$$\mu(t) = f(t) + l(-s^*)\int_0^t \frac{\tau^{-s^*-1}}{t^{-s^*}}f(\tau)d\tau. \qquad (18)$$

For a case $\lambda \in \mathbb{R}$ the statement takes place.

Proposition 2. If setpoint $\lambda_1 = Re\,\lambda$ on the half $s_1 = Re\ s < 1-\alpha$ function $1-\lambda_1\tilde{k}(s_1)$ has a single zero at $\lambda_1 > 0$ and has no zeros at $\lambda_1 < 0$, on the half $s_1 > 1-\alpha$ for any values λ_1 this function has a countable number of zeros.

Theorem 2. For any function f (t) an inhomogeneous integral equation (1) has a solution of class (2):

$$\mu(t) = f(t) + l(-s^*)\int_0^t \frac{\tau^{-s^*-1}}{t^{-s^*}}f(\tau)d\tau + Ct^{-s^*}\,, \quad Re\,\lambda > 0;$$

$$\mu(t) = f(t) + \int_0^t \sum_{k=1}^\infty l(s_k^0)\frac{\tau^{s_k^0-1}}{t^{s_k^0}}f(\tau)d\tau\,, \quad Re\,\lambda < 0.$$

Indeed, for $Re\,\lambda > 0$ we have

$$e^{-t}\mu(t) = e^{-t}f(t) + l(-s^*)\int_0^t e^{-(t-\tau)}\left(\frac{\tau}{t}\right)^{-s^*}e^{-\tau}f(\tau)\frac{d\tau}{\tau} + Ce^{-t}t^{-s^*}.$$

We need to show that the integral term belongs $L_1(\mathbb{R}_+)$. It follows from inequality

$$\left|\int_0^t e^{-(t-\tau)}\left(\frac{\tau}{t}\right)^{-s^*}e^{-\tau}f(\tau)\frac{d\tau}{\tau}\right| \leq \int_0^t \left(\frac{\tau}{t}\right)^{-s^*}|e^{-\tau}f(\tau)|\frac{d\tau}{\tau}.$$

The same holds for the case $Re\,\lambda < 0$.

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