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ON THE DISCRETE PART OF THE SPECTRUM OF AN ELLIPTIC OPERATOR

Abstract: *The article "On the discrete part of the spectrum of an elliptic operator" examines the study of the nature of the spectrum, which consists in studying a set of spectral points depending on the behavior of coefficients in a differential operation, the type and nature of boundary value problems. The question of the structure of the spectrum of differential operators is important. Special attention is paid to the construction of functional calculus and the proof of the spectral theorem in its various formulations.*

Key words: *operator, discreteness of the spectrum, differential operation, set, inequality.*

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Introduction

Let $\mathcal{L}$ be the operator generated by the differential operator$(-\Delta)^m + a(x)$ on the set $C_0^\infty(\Omega)$. Conditions are given for an unlimited domain of Ω and potential $a(x)$ ensuring the discreteness of the spectrum of the operator $\mathcal{L}$, $\Delta -$ the Laplace operator.

The paper investigates the discreteness of the spectrum of the operator $\mathcal{L}$ on the set $C_0^\infty(\Omega)$. Here $\Omega \subset \boldsymbol{R^n}$ is an unbounded domain, which can be tapering at infinity in a certain way, $\boldsymbol{R^n}$- is the Euclidean space $a(x) -$ is a measurable function.

Definition.

Private function of the operator L is called an element of the Sobolev space $\mathring{w}_2^m(\Omega)$ satisfying the equality

$$\int_\Omega \big(\nabla_m\varphi(x)\nabla_m U(x)\big)dx + \int_\Omega a(x)U(x)\varphi(x)dx = \lambda\int_\Omega U(x)\varphi(x)dx$$

for all functions $\varphi(x)$ rom the set $C_0^\infty(\Omega)$.

where: - $\mathring{w}_2^m(\Omega)$ - is the closure of the set of infinitely differentiable finite functions according to the norm of the Sobolev space $W_2^m(\Omega)$.

The discrete spectrum of an operator $\mathcal{L}$ is the set of its eigenvalues. The study of the nature of the spectrum consists in studying the set of points of the spectrum depending on the behavior of the coefficients in the differential operation, the type and nature of boundary value problems. The question of the structure of the spectrum of differential operators is also important for applications, for example: in quantum mechanics, the eigenvectors corresponding to the points of the discrete spectrum are called bound states, and the points of the discrete spectrum themselves are called bound state energies. Many works have been devoted to the study of the structure of the spectrum of elliptic operators [1]. We note some of them, in which significant results were obtained by M.S. Burman [2], I. M. Glazman [3], G. V. Rosenblum [4] M., Fefferman [5], P. Kerman and E. Sawyer [6]. V.G. Mazya obtained accurate results on the discreteness conditions of the negative spectrum in terms of capacitance for both the Schrodinger operator and the semi-harmonic operator [7, 8, 9]. An estimate of the lower edge of the continuous part of the spectrum, the perturbed semiharmonic operator A, is obtained in terms of capacitance.Toraev [10]. Yu.V. Egorov and V. A. Kondratiev obtained non-improving estimates for the negative spectrum of the Schrodinger operator, and then for the general elliptic operator of arbitrary order [4, 12].

Let's introduce the notation

$$\Omega_{r_0} = \{x \in \Omega : |x| > r_0\},$$

$$S_R = \{x \in \Omega : |x| = R\}$$

The continuous part of the spectrum of the operator $\mathcal{L}$ is denoted by and the domain of the operator $\mathcal{L}$ is denoted by $D(\mathcal{L})$. $P_d(b)$ – is a cube with an edge d parallel to the coordinate axes centered at point b.

Let T be the operator generated by the operation $(-\Delta)^m, D(T) = D(\mathcal{L})$) and K is the multiplication operator by the function $a(x)$. Then

$$\mathcal{L} = T + K.$$

First, let's give one well-known fact from [10].

The lemma. If $\lim_{|x|\to\infty} a(x) = 0$, then the operator K is completely continuous with respect to T.

Let $\mathcal{L}_1$ be the operator generated by the differential expression

$$(-\Delta)^m + a_1(x)$$

on the set $C_0^\infty(\Omega)$.

A consequence of the lemma: if a $a_1(x) = a(x)$ for $x \in \Omega_{r_0}$, then $C(\mathcal{L}) = C(\mathcal{L}_1)$

Really, let B be the multiplication operator by $a_1(x) - a(x)$ $D(B) = D(\mathcal{L}_1)$ According to the lemma, the operator B is completely continuous with respect to $\mathcal{L}$, therefore

$$C(\mathcal{L}) = C(\mathcal{L} + B) = C(\mathcal{L}_1).$$

Let the numbers β_i be determined by the fact that

$$\sum_{i=0}^{m} \beta_{2i}\lambda^{2i} = \left[\lambda^2 + \left(\frac{n-2m}{2}\right)^2\right]\left[\lambda^2 + \left(\frac{n-2m+4}{2}\right)^2\right]\ldots$$
$$\ldots\left[\lambda^2 + \left(\frac{n+2m-4}{2}\right)^2\right] \quad (1)$$

the coefficients of which will be used below

When proving our theorems, we will use the inequality

$$(\mathcal{L}_1 U, U) \geq \int_\phi \int_{t_0}^{\infty} \left[\sum_{i=0}^{m} \beta_{2_i}\left(\frac{d^i V}{dt^i}\right)^2 + N\, exp(\gamma t)V^2 + a(t,\varphi)\exp(2m)\, t)V^2\right] d\varphi \quad (2)$$

from the work [13], here $t = \ln r$, $U = V(t)\exp\left(\frac{2m-n}{2}\right)t, \phi -$ the area of change $\varphi = (\varphi_1\varphi_{2,\ldots,}\varphi_{n-1}), d\varphi -$ is the surface element $|\varphi| = 1$, β_{2i} are determined from equality (1).

Theorem 1. Let Ω be such that the set $\Omega \cap S_R$ for a sufficiently large R consists of a set of domains D_i such that if D_i^* there is an image of D_i when mapping $x^1 = {}^x/_{|x|}$ to S_1, then the eigenvalues of the operator

$$\sum_{i=1}^{m-1}(-1)^e C_m^e B_e \delta^e + (-1)^m\, \delta^m$$

then the eigenvalues of the operator D_i^* for any i not less than $NR^\gamma, \gamma = const, N = const > 0$, where the numbers $B_l(l = 1,2, \ldots, m-1)$ are determined by the characteristic numbers of ordinary differential operators of order $2l$ with constant coefficients [13], $\delta -$ operator Beltrami.

If

$$a(x) \geq -\beta_0|x|^{-2m} - \frac{\beta_2}{4}(|x|^m ln|x|)^{-2} - \beta N|x|^{\gamma-2m}$$

for some $0 < \beta < 1$, then the spectrum of the operator $\mathcal{L}$ is discrete, the numbers $\beta_0, \beta_2,$ are determined from formula (1).

Proof.

According to the corollary of the lemma, it is enough to consider the region Ω_{r_0}, where r_0 is large enough.

Since

$$\int_\phi \int_{t_0}^{\infty} \sum_{i=0}^{m} \beta_{2i}\left(\frac{d^i V}{dt^i}\right)^2 dt \geq 0,$$

then from inequality (2) using the conditions of the theorem we have

$(\mathcal{L}U,U) \geq \int_{\phi} \int_{t_0}^{\infty} \left[\left(\beta_2\left(\frac{dV}{dt}\right)^2 - \frac{1}{4t^2}\beta_2 V^2\right) \mp\right.$
$\beta_0 V^2 + N\exp(\gamma t)\,V^2 - (\beta_0|x|^{-2m} -$
$\left.-\beta N|x|^{\gamma-2m})\exp(2mt)\,V^2\right]dt\,d\varphi$ (3)

According to the well-known inequality of [14, c.13]

$$\int_{t_0}^{\infty}\left(\frac{dV}{dt}\right)^2 dt \geq \frac{1}{4}\int_{t_0}^{\infty}\frac{V^2}{t^2}dt$$

fair $\forall\, V \in C_0^{\infty}(R^1)$. Therefore, taking into account the performed substitutions in inequality (2), inequality (3) can be written in the form

$$(\mathcal{L}U,U) \geq \int_{\Omega_{r_0}} [\beta_0|x|^{-2m} + N|x|^{\gamma-2m} - \beta_0|x|^{-2m} - \beta N|x|^{\gamma-2m}]U^2 dx.$$

Let the statement of the theorem be incorrect and the number $M > 0$ is a point of a continuous spectrum. Let's denote $\mathcal{L}_2 = \mathcal{L} - M$. From the last inequality we will have

$$(\mathcal{L}_2U,U) \geq \int_{\Omega_{r_0}} \left[\left(\frac{1-\beta}{2}\right)N|x|^{\gamma-2m} + \frac{1-\beta}{2}N|x|^{\gamma-2m} - \beta_0|x|^{-2m} - M|x|^{-2m}\right]U^2 dx >$$

since $\gamma > 2m$ $0 < \beta < 1$, and this contradicts our proposal that $\mathrm{M} \in C(\mathcal{L})$, which proves the theorem, according to the corollary of the lemma.

Theorem 2. Let the domain Ω be the same as in Theorem 1 and

$q(x) = a(x) + \beta_0|x|^{-2m} + \sigma N|x|^{2-2m}$ (4)

for some $0 < \alpha < 1,\ 0 < \sigma < \alpha$.

If

1.for a fixed d>0, for $|x| \to \infty$

$\int_{P_d(b)}|q_-|(y)|y^{2m-\gamma}|\,dy \to 0, P_d(\mathrm{b}) \cap \Omega_{r_0} = \emptyset.$ (5)

2. for $d \to 0$ uniformly relative to

$$\int_{P_d(b)} |q_-(y)|\,|y^{2m-\gamma-1}||x-y|^{1-n}\,dy \to 0, \quad P_{d(b)} \cap \Omega_{r_0} \neq \emptyset, \qquad (6)$$

$\forall (x,y) \epsilon P_d\,(b) > r_0,\ \ r_0$ – is quite large.

then the spectrum of the operator $\mathcal{L}$ is discrete.

Proof.

It is not difficult to see the fairness of inequality

$\frac{|x^i|}{|y^i|} \geq c_0 \quad c_0 = const > 0, \qquad i = const$ (7)

Applying the formula of the integral representation of the Sobolev function through its first derivatives (see [15], p. 436) to the function U^2 *(x)*, we obtain

$$U^2(x) \leq C_1\left[d^{-n}\int_{P_d(b)} U^2(y)dy + \int_{P_d(b)} |U(y)||\nabla U||x-y|^{1-n}dy\right] \qquad (8)$$

where $x\epsilon P_d(b)$ and the constant C_1 does not depend on *d*. If r_0 - is large enough and $0 < d < 1, |b| > r_{0,}$ then due to inequality (7) of (8) we will have

$$U^2(x) \leq C_2\left[d^{-n}|x|^{2m-\gamma}\int_{P_d(b)} |y|^{\gamma-2m}\,U^2(y)dy + \right.$$
$$\left. + |x|^{2m-\gamma-1}\int_{P_d(b)} |y|^{\gamma-2m+1}|U(y)|\,|\nabla U||x-y|^{1-n}dy\right] \qquad (9)$$

will cover the area Ω_{r_0} к with a cubic lattice. with a cubic latticeP_d the inequality (9) is valid

Integrating this inequality with weight$|q\,(x)|$ for each non -empty intersection with Ω_{r_0} for any cube of the lattice, we get

$$\int_{P_d} |q_-(x)|U^2dx \le C_2 d^{-n} \int_{P_d} |q_-(x)|\,|x|^{2m-\gamma}dx \int_{P_d} U^2(y)|y|^{\gamma-2m}\,dy +$$

For a given $\varepsilon > 0$, by virtue of condition (6), we take the number d so small that

$$\frac{C_2}{2}\int_{P_d} |q_-(x)|\,|x|^{2m-\gamma-1}|x-y|^{1-n}dx < \varepsilon.$$

Fixing d by virtue of condition (5), we take the number r_0 so large that

$$C_2 d^{-n}\int_{P_d} |q_-(x)|\,|x|^{\gamma-2m}\,dx < \varepsilon.$$

Then

$$\int_{P_{d_0}} |q_-(x)|\,U^2(x)dx \le \varepsilon \int_{P_d} (|\nabla U|^2|y|^{2-2m+\gamma} + |y|^{\gamma-2m}\,U^2(y))dy$$

Summing up this inequality over all cubes of the lattice, we obtained an estimate

$$\int_{\Omega_{r_0}} |q_-(x)|U^2(x)dx \le \varepsilon \int_{\Omega_{r_0}} (|\nabla U|^2|x|^{2+\gamma-2m} + U^2(x)|x|^{\gamma-2m})\,dx. \qquad (10)$$

We Have

$$(\mathcal{L}U,U) = \big((T+K)U,U\big) = \big((1-\alpha)T+\alpha T\big)+K)U,U\big) = \big((1-\alpha)TU,U\big) + \big((\alpha T+K)U,U\big),$$

where $0<\alpha<1$.

Since

$$(\mathcal{L}U,U) \ge (1-\alpha)TU,U) + (\alpha TU,U) +$$

$$+\int_{\Omega_{r_0}} |[-\mu_0|x|^{-2m}U^2 - \sigma N|x|^{\gamma-2m}U^2 - |q_-(x)|U^2]|dx.$$

Hence, by virtue of inequality (10), we will have

$$(\mathcal{L}U,U) \ge \big((1-\alpha)TU,U\big) + (\alpha TU,U) + \int_{\Omega_{r_0}} [\alpha\mu_0|x|^{-2m}\,U^2 +$$

Consider the expression

$$((1-\alpha)TU,U) - \varepsilon\int_{\Omega_{r_0}} (|\nabla U|^2|x|^{2-2m+\gamma})dx.$$

Using inequality (2), it can be proved that

$$((1-\alpha)TU,U) \ge C_4 \int_{\Omega_{r_0}} |x|^{\gamma-2m+2}\,(\nabla U)^2dx, \qquad C_4 = const$$

Then moving from t to x and from V to U by virtue of inequality (11) will have

$$(\mathcal{L}U,U) \ge \int_{\Omega_{r_0}} [\alpha\beta_0|x|^{-2m} + \alpha N|x|^{\gamma-2m} - \beta_0|x|^{-2m} -$$

$$-\sigma N|x|^{\gamma-2m} - \varepsilon|x|^{\gamma-2m}]U^2dx$$

or

$$\int_{\Omega_0} (\alpha\beta_0|x|^{-2m} - \beta_0|x|^{-2m} + \alpha N\,|x|^{\gamma-2m} - \beta N|x|^{\gamma-2m})U^2dx \ge$$

$$\geq \int_{\Omega_0} \left((\alpha-1)\beta_0 |x|^{-2m} + \frac{\alpha-\beta}{2} N|x|^{\gamma-2m} + \frac{\alpha-\beta}{2}|x|^{\gamma-2m} \right) U^2 dx \geq$$

$$\geq \int_{\Omega_0} \frac{\alpha-\beta}{2}|x|^{\gamma-2m} U^2 dx.$$

Since $\alpha > \beta$ and $\gamma > 2m$. Let the number $M > 0$ be a point of a continuous spectrum, then

$$(\mathcal{L}_2 U, U) \geq \int_{\Omega_0} \left(\frac{\alpha-\beta}{2}|x|^{\gamma-2m} - M|x|^{-2m} \right) U^2 dx \geq C_4 \parallel U \parallel^2_{\mathcal{L}_2(\Omega_0)},$$

$C_4 = const > 0$, and this contradicts our assumption that M is a point of a continuous spectrum. Since according to the corollary of Lemma $C(\mathcal{L}_2) = C(\mathcal{L})$.

The theorem has been proved.

Consequence.

Let all the conditions of Theorem 1 be fulfilled and

$$\gamma = 2m, a(x) \geq -\beta_0 |x|^{-2m} - \frac{\beta_2}{4}(|x|^m ln|x|)^{-2} - \sigma N.$$

For some $\sigma < \frac{1}{2}$, then on the$(-\infty, \sigma N)$ the spectrum of the operator $\mathcal{L}$ is exhausted by the eigenvalues of finite multiplicity, the only possible limit point $\lambda = \sigma N$.

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