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INTEGRATION BY PARTS FORMULAS AND FORMULATION OF FEYNMAN PATH INTEGRAL

***Abstract**: This paper aimed at investigating the Integration by Parts Formulas and Rotationally Invariant Sobolev Calculus on Free Loop Spaces of a manifold mechanics. This formulation is very familiar to us and well known to be useful. But its rigorous meaning is given little except for special cases.*
***Key words**: Sobolev calculus, loop space, Laplacian, Feynman path integral, Lagrangian function, Schrödinger equation, Continuously uniformly.*
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Introduction

In [2] the project of understanding some of the analytical properties of Chen forms by discussing the L^p-theory of Chen forms is started. Then Δ is a uniformly elliptic second order differential operator. let Ω_x be the space of loops in $\mathbb{R}^d$ which start at x and return to x after a time period 1. Let $dp_1^{x,x}$ be the Brownian bridge measure on Ω_x.

Let H $(\mathbb{R}^d)$ be the Hilbert space of paths $H : [0,1] \to \mathbb{R}^d$ which are absolutely continuous with square integrable derivative, equipped with the norm

$$\|H\|^2 = \int_0^1 |H(t)|^2 d\,t + \int_0^1 \left|\frac{dH(t)}{dt}\right|^2 dt\,.$$

Now, as in [2], for each loop $w \in \Omega$ we consider the Hilbert space H $_w$ consisting of the vector fields along the loop w of the form $\mathrm{X} = \tau H$ with periodicity assumption where $(\tau H)_t = \tau_t(w)H_t$ with $\tau_t(w)$ being stochastic parallel transport along the loop w. These Hilbert spaces form a measurable field of tangent Hilbert spaces on the loop space and they will play the role of the tangent spaces of the loop space.

These Hilbert spaces form a measurable field of tangent Hilbert spaces on the loop space and they will play the role of the tangent spaces of the loop space.

The basic tool in setting up this Sobolev calculus is integration by parts formulas. In [3] we use the Peano approximation to the diffusion associated to Δ, which leads to suitable finite dimensional approximations to the Bismut measure μ. In [3], we define a Skorohod anticipative integral (see [4]) and an Ornstein−Uhlenbeck operator L such that these functions belong to the domain of L^p for each $p > 0$.

Let M be a compact Riemannian manifold of dimension d and let $P(M)$ be the space of all continuous maps $w : [0,1] \to M$. Let ν be the measure $\mathrm{P}_1(x,y)\, d\mathrm{P}_1^{x,y}\, dx\, dy$ on P(M), where $\mathrm{P}_1(x,y)$ is the heat kernel of M and $d\mathrm{P}_1^{x,y}$ is the Brownian bridge measure on $\mathrm{P}_{x,y}$(M), the space of paths joining x to y. We use the notation $\mathrm{E_P}$ for expectations computed with respect to this measure.

Let $V(t,x) \in \mathrm{R}\,, A(t,x) = (A_1, \dots, A_d) \in \mathrm{R}^d (x \in R^d, t \in [0,T])$ be the electromagnetic potentials, which are defined from

$$E_j = -\frac{\partial A_j}{\partial t} - \frac{\partial V}{\partial x_j} (j = 1, \dots, d),$$

$$d\left(\sum_{j=1}^{d} A_j\, dx_j\right) = \sum_{1 \le j < k \le d} B_{jk}\, dx_j \wedge dx_k \quad \text{on } \mathrm{R}^d \ (1.1)$$

for the electric strength $E(t,x) = (E_1, \dots E_d)$ and the magnetic strength tensor $\left(B_{jk}(t,x)\right)_{1\le j<k\le d}$. Then the Lagrangian function, the Hamiltonian, and the Schrödinger equation are given by

$$\mathcal{L} = (t,x,\dot{x}) = \frac{m}{2}|\dot{x}|^2 + (\dot{x}\cdot A - V), x^{\cdot} \in \mathrm{R}^d, \qquad (2.1)$$

$$H(t) = \frac{1}{2m}\sum_{j=1}^{d}\left(\hbar D_{x_j} - A_j\right)^2 + V\left(D_{x_j} = \frac{\partial}{i\partial x_j}\right), \qquad (3.1)$$

and

$$i\hbar\frac{\partial}{\partial t}u(t) = H(t)u(t) \quad (t\in[s,T]),\ \ u(s) = f \quad (4.1)$$

respectively. We sometimes write the solution $u(t)$ of (4.1) as $U(t,s)f$. For a multiindex

$\alpha = (\alpha_1, \dots, \alpha_d)$ we write

$\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_d)^{\alpha_d}$ and

$|\alpha| = \sum_{j=1}^{d}\alpha_j$.

Feynman in [6,7] expressed the solution of (4.1) in the integral form, which is called the Feynman path integral,

$$U(t,s)f = \frac{1}{\mathrm{N}}\int_{\Gamma_x} e^{i\hbar^{-1}S(\gamma)} f(\gamma(s))\,\mu(d\gamma), \quad (5.1)$$

where $\mu(d\gamma)$ is a uniform measure on $(\mathrm{R}^d)^{[s,t]}$ and N is a normalization factor.

In [8,9,10,11,12], etc., equations with the potentials

$$V = \sum_{j,k=1}^{d} a_{jk}x_jx_k + \sum_{j=1}^{d} b_jx_j + \int_{\mathrm{R}^d} e^{ix\cdot y}\,\nu(dy),$$

$$A_l = \sum_{j=1}^{d} c_{lj}x_j + d_l \quad (l = 1,\dots,d)$$

were studied, where a_{jk}, b_j, c_{lj} and d_l are constants and $\nu(dy)$ is a complex measure of bounded variation on R^d .

We study the formulation of the Feynman path integral through broken line paths. This formulation is very familiar to us and well known to be useful. We show rigorously for some class of potentials that this formulation is well defined and that this Feynman path integral gives the probability amplitude, i.e., the solution of the Schrödinger equation.

Let Δ be the subdivision of [0, t] above and $x^{(j)} \in \mathrm{R}^d(\mathrm{j} = 0,1,\dots,\mathrm{n}-1)$. We denote by $\gamma_\Delta = \gamma_\Delta(x^{(0)}, x^{(1)}, \dots, x^{(n-1)}, x) \in (\mathrm{R}^d)^{[0,t]}$ the broken line path joining

$\left(t_j, x^{(j)}\right)\ \left(j = 0,1,\dots,n, x^{(n)} = x\right)$. Set

$$\mathrm{C}(\Delta)f = \prod_{j=1}^{n}\sqrt{\frac{m}{2\pi i\hbar\left(t_j - t_{j-1}\right)}}^{\,d}\int_{\mathrm{R}^d}\dots\int_{\mathrm{R}^d} e^{i\hbar^{-1}S(\gamma_\Delta)} f\left(x^{(0)}\right)dx^{(0)}dx^{(1)}\dots dx^{(n-1)}. \qquad (6.1)$$

II. Preliminaries

To define a connection ∇ on the tangent spaces to $P(\mathrm{M})$ suppose X and Y are vector fields on $P(\mathrm{M})$, that is, sections of the field of Hilbert spaces $\mathscr{H}$. Now $Y = \tau K$ and we can write out K in components

$$K(w) = \sum_{i=1}^{d} k^i(w)V_i(w_0),$$

where the K^i are functions on $P(\mathrm{M})$ and the V_i are vector fields on M. Now define $\nabla_\mathrm{X}\mathrm{K}$ by the formula

$$(\nabla_\mathrm{X}\mathrm{K})_w = \sum_{i=1}^{d}\langle dk^i, \mathrm{X}\rangle_w V_i(w_0) + k^i(w)\left(\nabla_{\mathrm{X}_0}V_i\right)_{w_0}, \qquad (1.2)$$

and define $\nabla_\mathrm{X}Y$ by $\nabla_\mathrm{X}Y = \tau\nabla_\mathrm{X}\mathrm{K}$.

Theorem (1.2): For any smooth function $f: \mathrm{M}^k \to \mathbb{R}$ and any set of times $u = (u_1, \dots, u_k)$ with $0 < u_1 < \dots < u_k < 1$ let F be the function on $P(\mathrm{M})$ given by $F(w) = f\big(w(u_1), \dots, w(u_k)\big)$. Then we have the following integration by parts formula:

$$E_P[\langle dF, \mathrm{X}\rangle] = E_P\left[F\left(div\,\mathrm{X}_0(w_0) + \int_0^1\langle\tau_s h_s, \delta w_s\rangle + \frac{1}{2}\int_0^1\langle S_{\tau_s H_s}, \delta w_s\rangle\right)\right], \qquad (2.2)$$

where X is a vector field on $P(\mathrm{M})$, S is the Ricci tensor of M and δ the Itô integral.

Lemma (2.2): Let S and T be two stopping times. There exist semi-martingale processes A and B such that $\mathrm{A_S}$ and $\mathrm{B_s}$ are smooth functions of $\mathrm{w_s}$ with values in forms such that $\sum_{\mathrm{S}<\mathrm{t_i}<T}\mathrm{div}\,\mathrm{X}^{\mathrm{N}}_{\mathrm{t_i}}$ converges as $\mathrm{N}\to\infty$ to

$$\int_S^T\langle A_s(X_s), \delta w_s\rangle + \int_S^T B_s(X_s)\,ds.$$

Proof: We can suppose that over the interval $[S,T]$ the process lies in a local chart for M. We work in normal coordinates. If τ_s^N is parallel transport along the geodesic joining $w_{t_{i-1}}$ *to* w_{t_i} for $i \ge 1$ *it* follows that

$$d\tau_s^N = -\Gamma_{\tau_s^N}\left(w_{t_{i-1}} + s\left(w_{t_i} - w_{t_{i-1}}\right)\right)\left(w_{t_i} - w_{t_{i-1}}\right)ds, \qquad \tau_0^N = \tau_{i-1}^N, \qquad (3.2)$$

where the Christoffel matrix Γ is zero at $w_{t_{i-1}}$, is the analogue of the cancellation which appears in ref. [3] and we deduce that

$$div\, X_{t_i}^N = A'\left(w_{t_{i-1}}, X_{t_{i-1}}^N + (t_{i+1} - t_i)\tau_{i-1}^N h_{t_{i-1}}\right)\left(w_{t_i} - w_{t_{i-1}}\right) + B'\left(w_{t_{i-1}}, X_{t_{i-1}}^N + (t_{i+1} - t_i)\tau_{i-1}^N h_{t_i-1}\right)\left(w_{t_i} - w_{t_{i-1}}\right)^2 + o\left(\left(w_{t_i} - w_{t_{i-1}}\right)^3\right) + o(t_i - t_{i-1}), \qquad (4.2)$$

and the proof of Lemma (2.2) is completed in the same way as the proof of Lemma(1.1.4)[1].

From both previous lemmas we see that

$$E_P[\langle dF, X\rangle] = E_P\left[F\left(div\, X_0 + \int_0^1 \langle A(X_s), \delta w_s\rangle + \int_0^1 \langle B(\tau_s h_s), \delta w_s\rangle + \int_0^1 C(X_s)\, ds + \int_0^1 D(\tau_s h_s)\right)\right]. \qquad (5.2)$$

If $X_0 = 0$ we know that the sum of the above integrals is equal to

$$\int_0^1 \langle \tau_s h_s, \delta w_s\rangle + \frac{1}{2}\int_0^1 \langle S_{\tau_s H_s}, \delta w_s\rangle.$$

The formula for $E_P[\langle dF, X\rangle]$ can be extended to all previsible h_s bounded in L^2.

Moreover A, B, C, D have the same shape as in [9]. However, if $\bar{A}, \bar{B}, \bar{C}$, and $\bar{D}$ are processes of this particularly nice type such that for all $X_s = \tau_s H_s$, where $H_s = \int_0^s h_u\, du$ with h previsible and bounded, we have

$$\int_0^1 \langle \bar{A}(X_s), \delta w_s\rangle + \int_0^1 \langle \bar{B}(\tau_s h_s), \delta w_s\rangle + \int_0^1 \bar{C}(X_s)\, ds + \int_0^1 \bar{D}(\tau_s h_s)\, ds = 0,$$

then it follows that $\bar{A} = \bar{B} = \bar{C} = \bar{D} = 0$.

If $\bar{B}$ is not zero we can choose a stopping time $T < 1$, with probability strictly positive, and a deterministic process $\tilde{h}_s$ with $\tilde{h}^{(k)}(0) \neq 0$ such that on the condition that $F_T < 1$, if we take $h_s = h_{T+s}$, the sum of the four integrals in the above formula has a density such that from Malliavin calculus we deduce from the same type of argument that $\bar{A} = 0$. For all X_s we have

$$\int_0^1 \bar{C}(X_s)\, ds + \int_0^1 \bar{D}(\tau_s h_s)\, ds = 0\,.$$

But $\bar{D}$ is a smooth function in w_s. We deduce that $\bar{D}(\tau_s)$ has bounded variation, and from stochastic calculus it is constant.

That implies $\bar{C} = 0$.

Proposition (2.3): Let V and A satisfy the assumptions of the Theorem. Then there exists a continuous function $q(t, s; x, w)$ in $0 \le s \le t \le T, x, \omega \in R^d$ satisfying $\left(\left|\partial_\omega^\alpha \partial_x^\beta p(x, w)\right| \le C_{\alpha\beta}\langle x; w\rangle^M,\quad x, \omega \in R^d\right)$ with an M so that for $f \in S$

$$\left\{i\hbar\frac{\partial}{\partial t} - H(t)\right\}\mathcal{C}(t,s)f = \sqrt{t-s}\sqrt{\frac{m}{2\pi i\hbar(t-s)}}^{\,d} \times \int e^{i\hbar^{-1}S\left(\gamma_{x,y}^{t,s}\right)} q\left(t, s; x, \frac{x-y}{\sqrt{(t-s)}}\right) f(y)dy, \quad 0 \le s < t \le T. \qquad (6.2)$$

Proof: By direct calculations we have from(3.1),

$$\left\{i\hbar\frac{\partial}{\partial t} - H(t)\right\}\mathcal{C}(t,s)f = -\sqrt{\frac{m}{2\pi i\hbar(t-s)}}^{\,d} \times \int e^{i\hbar^{-1}S\left(\gamma_{x,y}^{t,s}\right)}\left\{r_1(t,s;x,y) + \frac{i\hbar}{2m} r_2(t,s;x,y)\right\} f(y)dy, \qquad (7.2)$$

$$r_1 = \partial_t S\left(\gamma_{x,y}^{t,s}\right) + \frac{1}{2m}\sum_{j=1}^{d}\left\{\partial_{x_j} S\left(\gamma_{x,y}^{t,s}\right) - A_j(t,x)\right\}^2 + V(t,x), \qquad (8.2)$$

$$r_2 = \frac{dm}{t-s} - \Delta_x S\left(\gamma_{x,y}^{t,s}\right) + (\nabla . A)(t,x), \qquad (9.2)$$

where $\nabla . A = \sum_j \partial_{x_j} A_j$.

Set $\rho = t - s$. Using

$$S(\gamma_{x,y}^{t,s}) = \frac{m|x-y|^2}{2(t-s)} + \int_{\gamma_{x,y}^{t,s}} A.\,dx = \frac{m|x-y|^2}{2(t-s)} + (x-y).\int_0^1 A\big(s+\theta(t-s), y+\theta(x-y)\big)\,d\theta - (t-s)\int_0^1 V\big(s+\theta(t-s), y+\theta(x-y)\big)\,d\theta,$$

we have

$$\partial_{x_j}S(\gamma_{x,y}^{t,s}) - A_j(t,x) = \frac{m(x_j-y_j)}{\rho} + \int_0^1 A_j\left(s+\theta\rho, y+\theta(x-y)\right) - A_j(t,x)d\theta + \sum_k (x_k-y_k)$$
$$\times \int_0^1 \theta\frac{\partial A_k}{\partial x_j}\big(s+\theta\rho, y+\theta(x-y)\big)d\theta - \rho\int_0^1 \theta\frac{\partial V}{\partial x_j}\big(s+\theta\rho, y+\theta(x-y)\big)d\theta$$
$$= \frac{m(x_j-y_j)}{\rho} + \int_0^1 A_j\left(t-\theta\rho, x-\theta(x-y)\right) - A_j(t,x)d\theta$$
$$+\sum_k (x_k-y_k)\int_0^1 (1-\theta)\frac{\partial A_k}{\partial x_j}\big(t-\theta\rho, x-\theta(x-y)\big)d\theta$$
$$-\rho\int_0^1 (1-\theta)\frac{\partial V}{\partial x_j}\big(t-\theta\rho, x-\theta(x-y)\big)d\theta,$$

and so by the Taylor expansion

$$\partial_{x_j}S(\gamma_{x,y}^{t,s}) - A_j(t,x) = \frac{m(x_j-y_j)}{\rho} - \frac{1}{2}\sum_l \frac{\partial A_j}{\partial x_l}(t,x)(x_l-y_l) + \frac{1}{2}\sum_k \frac{\partial A_k}{\partial x_j}(t,x)(x_k-y_k) + \rho q_1\left(t,s;x,\frac{x-y}{\sqrt{\rho}}\right). \quad (10.2)$$

It follows from

$$-\sum_{j,l}(\partial_{x_l}A_j)(x_j-y_j)(x_l-y_l) + \sum_{j,k}\left(\partial_{x_j}A_k\right)(x_j-y_j)(x_k-y_k) = 0$$

that

$$\frac{1}{2m}\sum_{j=1}^{d}\left\{\partial_{x_j}S(\gamma_{x,y}^{t,s}) - A_j(t,x)\right\}^2 = \frac{m|x-y|^2}{2\rho^2} + \sqrt{\rho}q_2\left(t,s;x,\frac{x-y}{\sqrt{\rho}}\right). \quad (11.2)$$

The same arguments show that

$$\partial_t S(\gamma_{x,y}^{t,s}) = -\frac{m|x-y|^2}{2\rho^2} - V(t,x) + \sqrt{\rho}q_3\left(t,s;x,\frac{x-y}{\sqrt{\rho}}\right), \quad (12.2)$$

$$\Delta_x S(\gamma_{x,y}^{t,s}) = \frac{dm}{\rho} + (\nabla.A)(t,x) + \sqrt{\rho}q_4\left(t,s;x,\frac{x-y}{\sqrt{\rho}}\right). \quad (13.2)$$

Inserting (11.2) – (13.2) into (7.2) – (9.2), we can complete the proof.

III. Claims

Corollary (1.3): Assume
$(|\partial_x^\alpha A_j(t,x)| \le C_\alpha < x >^{-(1+\delta)},\ |\alpha| \ge 2,)$
and
$(|\partial_x^\alpha V(t,x)| \le C_\alpha, |\alpha| \ge 2,\ (t,x) \in [0,T] \times R^d.)$.

Then we have: (i)
$$\left|\partial_{x+\varepsilon}^{\alpha}\partial_x^{\beta}\phi(s+\varepsilon,s;x,x+\varepsilon)\right| \le C_{\alpha,\beta},$$
$$|\alpha+\beta| \ge 2,\quad \varepsilon \ge 0,\ x, x+\varepsilon \in R^d. \quad (1.3)$$

(ii) There exist constants $\rho_0 > 0$ *and* $\kappa > 0$ *such that*

$$\inf_{0\le\varepsilon\le\rho_0, x, x+\varepsilon} det\frac{\partial^2\phi}{\partial(x+\varepsilon)^2}(s+\varepsilon,s;x,x+\varepsilon) \ge \kappa, \quad (2.3)$$

where $\partial^2\phi/\partial(x+\varepsilon)^2$ is the Hessian in $(x+\varepsilon)$.

Proof: Let $|\alpha| \geq 1$. Then we have from$\left(\left|\partial_x^\alpha A_j(t,x)\right| \leq C_\alpha < x >^{-(1+\delta)},\ |\alpha| \geq 2,\right)$,

$$\left|\partial_x^\alpha A_j(s+\varepsilon,x) - \partial_x^\alpha A_j(s+\varepsilon,0)\right| \leq Const. \int_0^1 \frac{|x|}{\langle\theta x\rangle^{1+\delta}}\, d\theta \ \leq\ Const. \int_0^\infty \frac{1}{\langle\theta\rangle^{1+\delta}}\, d\theta < \infty,$$

and hence

$$\left|\partial_x^\alpha A_j(s+\varepsilon,x)\right| \leq C'_\alpha,\ |\alpha| \geq 1, (s+\varepsilon,x) \in [0,T] \times R^d. \tag{3.3}$$

In the same way we have for $|\alpha| \geq 2$,

$$\begin{aligned}\left|\sqrt{\rho}(x+\varepsilon)\,.\int_0^1 (\partial_x^\alpha A)\left(s+\theta\rho, x-(1-\theta)\sqrt{\rho}(x+\varepsilon)\right)d\theta\right| &\leq Const. \int_0^1 \frac{\sqrt{\rho}|x+\varepsilon|}{\langle x-\theta\sqrt{\rho}(x+\varepsilon)\rangle^{1+\delta}}\,d\theta \\ &\leq\ Const. \int_0^\infty \frac{1}{\langle x-\theta\Omega\rangle^{1+\delta}}\,d\theta \qquad \left(\Omega = \frac{x+\varepsilon}{|x+\varepsilon|}\right) \\ &\leq\ Const. \int_{-\infty}^\infty \frac{1}{\langle\theta\rangle^{1+\delta}}\,d\theta\ = C_\alpha^{"} < \infty,\quad \varepsilon \geq 0\,, x \in R^d,\end{aligned} \tag{4.3}$$

where we used $|x-\theta\Omega| \geq |\theta - x.\Omega|$. The inequality(1.3) can be shown from assumptions $(|\partial_x^\alpha V(t,x)| \leq C_\alpha, |\alpha| \geq 2,\ \ (t,x) \in [0,T] \times R^d.), \left(\phi(t,s;x,w) = \frac{m}{2}|w|^2 + \sqrt{\rho}w.\int_0^1 A(s+\theta\rho, x-(1-\theta)\sqrt{\rho}w)d\theta - \rho\int_0^1 V(s+\theta\rho, x-(1-\theta)\sqrt{\rho}w)d\theta\,,\quad \rho = t-s.\right)$, (3.3) and (4.3). So can (2.3), because we have $\partial^2\phi/\partial(x+\varepsilon)^2 = (m/2)I_d + O(\varepsilon)$. I_d is the identity matrix.

Lemma (2.3): Set $B_{jk} = -B_{kj}$ for $1 \leq k < j \leq d$ and $B_{jj} = 0$ for $j = 1,2,\dots,d$. Then we have

$$\iint_\Delta d(A\cdot dX) = (x-y)\cdot(\Psi_1,\dots,\Psi_d), \tag{5.3}$$

$$\Psi_j = -(t-s)\int_0^1\int_0^1 \sigma_1 E_j\left(\tau(\sigma),\zeta(\sigma)\right)d\sigma_1 d\sigma_2$$

$$-\sum_{k=1}^d (z_k - x_k)\int_0^1\int_0^1 \sigma_1 B_{jk}\left(\tau(\sigma),\zeta(\sigma)\right)d\sigma_1 d\sigma_2\,. \tag{6.3}$$

Proof: We have by $\left(\tau(\sigma),\zeta(\sigma)\right) = (1-\sigma_2)\{(1-\sigma_1)(t,z)+\sigma_1(s,x)\} + \sigma_2\{(1-\sigma_1)(t,z)+\sigma_1(s,y)\}\ = \left(t-\sigma_1(t-s), z+\sigma_1(x-z)+\sigma_1\sigma_2(y-x)\right) \in R^{d+1}.$,

$$\iint_\Delta E_j dt\wedge dx_j = \int_0^1\int_0^1 E_j\left(\tau(\sigma),\zeta(\sigma)\right)det\frac{\partial(\tau,\zeta_j)}{\partial(\sigma_1,\sigma_2)}\,d\sigma_1 d\sigma_2 = (t-s)(x_j-y_j)\int_0^1\int_0^1 \sigma_1 E_j\,d\sigma_1 d\sigma_2, \tag{7.3}$$

$$\begin{aligned}\iint_\Delta B_{jk}dx_j\wedge dx_k &= \int_0^1\int_0^1 B_{jk}\,det\frac{\partial(\zeta_j,\zeta_k)}{\partial(\sigma_1,\sigma_2)}\,d\sigma_1 d\sigma_2 \\ &-\{(x_k-y_k)(x_j-z_j)-(x_j-y_j)(x_k-z_k)\}\int_0^1\int_0^1 \sigma_1 B_{jk}\,d\sigma_1 d\sigma_2,\end{aligned} \tag{8.3}$$

and hence from Lemma(1.2.7)[5],

$$\begin{aligned}\iint_\Delta d(A\cdot dX) &= -(t-s)\sum_j (x_j-y_j)\int_0^1\int_0^1 \sigma_1 E_j\,d\sigma_1 d\sigma_2 \\ &- \sum_{1\leq j<k\leq d}\{(x_k-y_k)(x_j-z_j)-(x_j-y_j)(x_k-z_k)\}\int_0^1\int_0^1 \sigma_1 B_{jk}\,d\sigma_1 d\sigma_2 \\ &= -(t-s)\sum_{j=1}^d (x_j-y_j)\int_0^1\int_0^1 \sigma_1 E_j\,d\sigma_1 d\sigma_2\ + \sum_{j,k=1}^d (x_j-y_j)(x_k-z_k)\int_0^1\int_0^1 \sigma_1 B_{jk}\,d\sigma_1 d\sigma_2.\end{aligned}$$

Thus Lemma (2.3) could be proved.

IV. Conclusion

Was defined Sobolev spaces and an Ornstein-Uhlenbeck operator on the loop space. We find some functionals which belong to all the Sobolev spaces. We study two versions of the Sobolev calculus on the free loop space which are both invariant under rotations of loops. We work on $\mathbb{R}^d$ with metric $\sum g_{ij}dx^i dx^j$, where the g_{ij} are smooth bounded functions. These Hilbert spaces form a measurable field of tangent Hilbert spaces on the loop space and they will play the role of the tangent spaces of the loop space. The basic tool in setting up this Sobolev calculus is integration by parts formulas.

In [13,14] gave the rigorous meaning of the Feynman path integral for a class of potentials, adopting the formulation through piecewise classical paths. In [15] this result was generalized for a wide class of potentials. We study the formulation of the Feynman path integral through broken line paths. This formulation is very familiar to us and well known to be useful.

Using the ideas in the theory of difference methods and the theory of pseudo-differential operators, we show rigorously for some class of potentials that this formulation is well defined and that this Feynman path integral gives the probability amplitude, i.e., the solution of the Schrödinger equation.

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