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Issue Article

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PROPAGATION OF AXISYMMETRIC SURFACE WAVES IN A VISCOELASTIC IN HALF-PLANE

***Abstract**: In this work, The amplitudes of displacements and stresses decreasing with depth are estimated for a) a gradient-elastic half-space; b) a half-plane consisting of a Kossera medium.[2,3] The mathematical formulation, solution methodology, and algorithm for the problem of wave propagation in long-layered flat viscoelastic mechanical systems were developed for the Kossera medium.[7,8]*

***Key words**: neosymmetric voltage, viscoelastic half-plane, surface wave, transcendental equation, voltage amplitude, phase velocity.*

***Language**: English*

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Introduction

The study of the propagation of surface waves began in past centuries. In 1885, the English scientist Lord Rayleigh (John William Street) theoretically showed that a surface wave can propagate along the flat boundary of a rigid elastic half-space if it is confined to a vacuum or a sufficiently rarefied medium (for example, air). He found that its amplitude decreases rapidly with depth. These results have achieved significant results in seismology. Therefore, conducting research was relevant [1,2,3]. These waves, called Rayleigh surface waves, are applied to different directions of practice depending on the frequency range. Later it was also discovered that Rayleigh waves in the low-frequency range (1 - 100 Hz) are the main type of waves observed during earthquakes. Therefore, they have been studied in detail in seismology for many years [4]. One of the fundamental laws of Rayleigh wave propagation in an elastic medium is its non-dispersion, i.e., the wave velocity does not depend on its frequency. This is constant for each material, reaching a transverse wave velocity of 0.87 - 0.96. The displacement vector has longitudinal and transverse components, the transverse wave component is always greater than the longitudinal wave component [5]. In 2000, V.V. Krylova studied the elastic vibrations emitted by trains and cars. The theory he created was experimentally confirmed in 1997-1998. On some sections of the high-speed train line in Sweden (Göteborg-Malme), the speed of Rayleigh waves was 45 m/s, and the train speed, moving at a speed of 160 km/h, was sufficient to observe the effect [6,7]. It should be noted that the existence of critical speeds of freight movement (train) is indicated in railway instructions, exceeding which creates bending waves, which was discovered in the first half of the 1980s.

Propagation of a characteristic wave in a viscoelastic half-plane consisting of a Kossera medium

If a characteristic wave propagates in a half-plane, then the wave propagation equation, as given [8], is reduced to the following form.

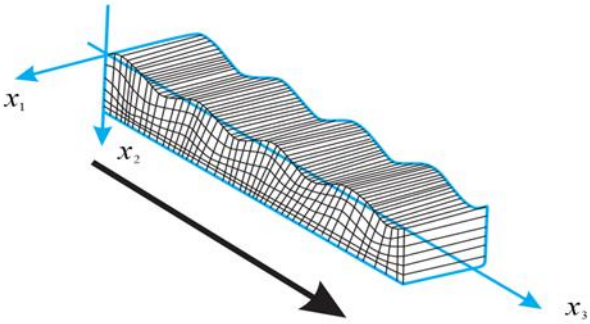

Figure 1. Rayleigh wave propagation

We obtain a system of ordinary differential equations with complex coefficients

$$(2\bar{\mu}_n + \bar{\lambda}_n) graddiv\bar{U}_n - (\bar{\mu}_n + \bar{\lambda}_n) rotrot\bar{U}_n + 2\alpha_n rot\bar{\Omega}_n + \rho_n\omega^2\bar{U}_n(z) = 0, \quad (1)$$

$$\bar{u}_n(x,z,t) = \bar{U}_n\{U_x(z), U_y(z), U_z(z)\}e^{i(kx-\omega t)}$$

$$\bar{\theta}_n(x,z,t) = \vec{\Omega}_n\{W_x(z), W_y(z), W_z(z)\}e^{i(kx-\omega t)} \quad (2)$$

If we substitute (2) into (1), we obtain the following system of ordinary differential equations:

$$(\bar{\alpha}_n + \bar{\mu}_n)\frac{d^2U_x(z)}{dz^2} + \left(\rho_n\omega^2 - k^2(\bar{\lambda}_n + 2\bar{\mu}_n)\right)U_x(z) + ik(-\bar{\alpha}_n + \bar{\mu}_n + \bar{\lambda}_n)\frac{dU_z(z)}{dz} - 2\bar{\alpha}_n\frac{dW_y(z)}{dz} = 0,$$

$$(\bar{\lambda}_n + 2\bar{\mu}_n)\frac{d^2U_z(z)}{dz^2} + (\rho_n\omega^2 - k^2(\bar{\alpha}_n + \bar{\mu}_n))U_z(z) + ik(-\bar{\alpha}_n + \bar{\mu}_n + \bar{\lambda}_n)\frac{d^2U_z(z)}{dz^2} - 2i\bar{\alpha}_n W_y(z) = 0,$$

$$(\bar{\alpha}_n + \bar{\mu}_n)\frac{d^2U_x(z)}{dz^2} + + \left(\rho_n\omega^2 - k^2(\bar{\lambda}_n + \bar{\mu}_n)\right)U_x(z) + 2\bar{\alpha}_n\frac{d^2U_z(z)}{dz^2} - 2ik\bar{\alpha}_n W_z(z) = 0,$$

$$(\bar{\gamma}_n + \bar{\varepsilon}_n)\frac{d^2U_x(z)}{dz^2} + (\rho_n\omega^2 - k^2(\bar{\gamma}_n + \bar{\varepsilon}_n) - 2\bar{\alpha}_n\frac{d^2U_x(z)}{dz^2} - 2ik\bar{\alpha}_n U_z(z) = 0,$$

$$(\bar{\gamma}_n + \bar{\varepsilon}_n)\frac{d^2U_x(z)}{dz^2} + (\rho_n\omega^2 - k^2(\bar{\gamma}_n + \bar{\varepsilon}_n) - 4\bar{\alpha}_n)W_y(z) + 2\bar{\alpha}_n\frac{d^2U_x(z)}{dz^2} - 2ik\bar{\alpha}_n U_z(z) = 0, \quad (3)$$

$$(\bar{\gamma}_n + \bar{\varepsilon}_n)\frac{d^2U_x(z)}{dz^2} + (\rho_n\omega^2 - k^2(2\bar{\beta}_n + \bar{\gamma}_n) - 4\bar{\alpha}_n)\frac{d^2U_x(z)}{dz^2} - 2\bar{\alpha}_n\frac{dU_y(z)}{dz} = 0,$$

$$(\bar{\gamma}_n + 2\bar{\beta}_n)\frac{d^2U_x(z)}{dz^2} + (\rho_n\omega^2 - k^2(\bar{\varepsilon}_n + \bar{\gamma}_n) - 4\bar{\alpha}_n)W_z(z) + ik(2\bar{\beta}_n - \varepsilon_n)\frac{dU_y(z)}{dz} - 2ik\bar{\alpha}_n U_y(z) = 0,$$

To simplify the problem-solving and calculation process, we reduce all dimensional quantities to dimensionless form. To solve the last obtained system of equations (3), we introduce the following new variables.

$$U_x(z) = ik\Phi_1(z) - \frac{d\Psi_1(z)}{dz},$$

$$U_x(z) = ik\Phi_2(z) - \frac{d\Psi_2(z)}{dz},$$
$$U_y(z) = \Omega_2(z),$$
$$W_y(z) = \Omega_1(z),$$
$$U_z(z) = ik\Psi_1(z) + \frac{d\Phi_1(z)}{dz},$$
$$W_z(z) = ik\Psi_2(z) + \frac{d\Phi_2(z)}{dz} \qquad (4)$$

If we substitute (4) into (3), we obtain a second-order ordinary differential system with complex coefficients.

$$\frac{d^2\Phi_1(z)}{dz^2} + \left(\frac{\omega^2}{C_1^2} - k^2\right)\Phi_1(z) = 0$$
$$\frac{d^2\Psi_1(z)}{dz^2} + \left(\frac{\omega^2}{C_3^2} - k^2\right)\Psi_1(z) + \frac{2}{B}\Omega_1(z) = 0,$$
$$\frac{d^2\Omega_1(z)}{dz^2} + \left(\frac{\omega^2}{C_4^2} - k^2 - \frac{4A^2B}{B-1}\right)\Omega_1(z) - \frac{2A^2B}{B-1}\frac{d^2\Psi_1(z)}{dz^2} + \frac{2k^2A^2B}{B-1}\Psi_1(z) = 0, \qquad (4)$$

$$\frac{d^2\Phi_2(z)}{dz^2} + \left(\frac{\omega^2}{C_5^2} - k^2 - \frac{4A^2BC_4^2}{(B-1)C_5^2}\right)\Phi_2(z) = 0,$$
$$\frac{d^2\Psi_2(z)}{dz^2} + \left(\frac{\omega^2}{C_3^2} - k^2 - \frac{4A^2B}{B-1}\right)\Psi_2(z) + \frac{2A^2B}{B-1}\Omega_2(z) = 0,$$
$$\frac{d^2\Omega_2(z)}{dz^2} + \left(\frac{\omega^2}{C_3^2} - k^2\right)\Omega_2(z) - \frac{2}{B}\frac{d^2\Psi_2(z)}{dz^2} + \frac{2k^2}{B}\Psi_2(z) = 0. \qquad (5)$$

The solutions of the above system of ordinary differential equations (5) are as follows.

$$\Phi_1(z) = E_0 e^{-\eta_0 z}$$
$$\Psi_1(z) = E_1 e^{-\eta_1 z} + E_2 e^{-\eta_2 z},$$
$$\Omega_1(z) = \frac{B}{2}\left\{E_1\left(x_1 - \frac{\omega^2}{C_3^2}\right)e^{-\eta_1 z} + E_2(x_2 - \frac{\omega^2}{C_3^2})e^{-\eta_2 z}\right\} \qquad (6)$$

similarly, the second group of solutions can be found

$$\Phi_2(z) = D_0 e^{-\xi_0 z}, \Psi_2(2) = D_1 e^{-\xi_1 z} + D_2 e^{-\xi_2 z},$$
$$\Omega_2(z) = \frac{B-1}{2A^2B}\left\{D_1(x_1 - \frac{\omega^2}{C_4^2} + \frac{4A^2B}{B-1})e^{-\xi_1 z}\right\} \qquad 7.\ a)$$

here

$$A = X_0\sqrt{\frac{\bar{\mu}}{B(\bar{\gamma}+\bar{\varepsilon})}},\ B = \frac{\bar{\alpha}+\bar{\mu}}{\bar{\alpha}}, C = \frac{\bar{\gamma}-\bar{\varepsilon}}{\bar{\gamma}+\bar{\varepsilon}}$$
$$C_1^2 = \frac{\lambda_0 + 2\mu_0}{\rho X_0^2 \omega_0^2}, C_3^2 = \frac{B}{B-1}C_2^2,$$
$$C_4^2 = \frac{\gamma_0+\varepsilon_0}{jX_0^2\omega_0^2}, C_5^2 = \frac{\gamma_0+2\beta_0}{jX_0^2\omega_0^2},$$

Where C_1 and C_2 are taken as the same as the concept of longitudinal and transverse wave propagation, C_4 and C_5 are called the Cossssera parametrs, C_3 is included to reduce the notation. Using the above $U_n(z), W_n(z)$- we can also write the solutions

$$U_x(z) = ikE_0 e^{-\eta_0 z} + E_1\eta_1 e^{-\eta_1 z} + E_2\eta_2 e^{-\eta_2 z}$$

$$U_y(z) = \frac{B-1}{2A^2B}\left\{D_1\left(x_1 - \frac{\omega^2}{C_4^2} + \frac{4A^2B}{B-1}\right)e^{-\xi_1 z} + D_2(x_2 - \frac{\omega^2}{C_4^2} + \frac{4A^2B}{B-1})e^{-\xi_2 z}\right\}$$
$$U_z(z) = \eta_0 E_0 e^{-\eta_0 z} + irE_1 e^{-\eta_1 z} + ikE_2 e^{-\eta_2 z}$$

$$W_z(z) = ikD_0 e^{\xi_0 z} + ikD_1 e^{-\xi_1 z} + ikD_2 e^{-\xi_2 z}, \quad (7,b)$$

$$W_z(z) = \frac{B}{2}\left\{E_1\left(x_1 - \frac{\omega^2}{C_3^2}\right)e^{-\eta_1 z} + E_2(x_2 - \frac{\omega^2}{C_3^2})\right\}$$
$$W_z(z) = ikD_0 e^{-\xi_0 z} + ikD_1 e^{\xi_0 z} + ikD_2 e^{-\xi_1 z} + ikD_2 e^{-\xi_2 z}$$

here

$$\eta_0 = \sqrt{k^2 - \frac{\omega^2}{c_1^2}},\ \xi_0 = \sqrt{k^2 + \frac{4A^2BC_4^2}{(B-1)C_5^2} - \frac{\omega^2}{c_5^2}},$$

$$\eta_1 = \xi_1 = \sqrt{k^2 - x_1}, \eta_2 = \xi_2 = \sqrt{k^2 - x_2}$$

$$x_{12} == \frac{C_3^2 + C_4^2}{2C_3^2C_4^2}\omega^2 - 2A^2 \pm \sqrt{\frac{(C_3^2 - C_4^2)^2}{4C_3^4C_4^4}\omega^4 \frac{2A^2(C_2^2C_3^2 - 2C_3^2C_4^2 + C_2^2C_4^2)}{C_2^2C_3^2C_4^4}\omega^2 + 4A^2}$$

The following volumetric wave propagation speeds are known for the Cossera medium

$$c_l^2 = \frac{\lambda + 2\mu}{\rho};\ c_{lr}^2 = \frac{\beta + 2\gamma}{\rho I};\ c_s^2 = \frac{\mu}{\rho};$$

$$c_{s\alpha}^2 = \frac{\mu + \alpha}{\rho};\ c_{sr}^2 = \frac{\gamma + \varepsilon}{\rho I}.$$

Based on the obtained solution, the problems of free wave propagation or stimulated (stationary or harmonic, unstable) wave propagation in the Cosser medium are solved.

$$det_1[D_{p1}(\omega, k)] = 0, det_2[D_{p2}(\omega, k)] = 0 \quad (8)$$

The dispersion equation (8) is obtained in analytical form. Since the medium is viscoelastic, the unknowns in the dispersion equation are complex quantities. The complex-parameter transcendental equation (8) is solved numerically using the Müller and Laplace methods. Below is a general algorithm for finding the dispersion relation and obtaining numerical results in structures consisting of a multilayer Cossera medium. Conclusions are drawn on the formulation, methodology, and algorithm of problem-solving. In the calculations, a Rzhanitsyn-Koltunov kernel with a three-parameter weak singularity was obtained for nuclear relaxation.

$$R(t) = Ae^{-\beta t}/t^{1-\alpha}$$

Algorithm for solving the dispersion relation in structures consisting of a multilayer Cossera medium

When studying the propagation processes of damping waves in a homogeneous medium with an elastic layer with plane-parallel interfaces, it is first necessary to determine the dispersion properties of these waves. Dispersion characteristics refer to phase and group velocities V_ϕ and V_g, Ω_I understood as the coefficients of wear over time. As is known V_ϕ and Ω_I.

The values are related to the value of the root of the variance equation.

$$\Delta(\omega, \xi) = 0 \quad (9)$$

Formulas

$$V_\phi = Im\omega(\xi), \Omega_I = -Re\omega(\xi), \Omega_I = -Re\omega(\xi),$$

where ω –complex frequency, ξ – wave number.

When solving the problems of propagation of stress waves in a viscoelastic medium, we arrive at frequency equations with complex coefficients and roots. Solving complex root frequency equations, which are often complex transcendental equations, is also difficult when using computers. In addition, it is necessary to take into account that to obtain the phase velocity or wavelength, the task must be repeated at different constant frequencies.

It is advisable to directly determine complex roots using Müller's methods. For complex roots, the Müller method simplifies calculations and provides convergence faster than the Barstou method if the roots are close to each other [9]. Müller's method uses quadratic interpolation, resulting in a form iteration:

$$Z^{[j+1]} = Z^{[j]} - (Z^{[j]} - Z^{[j-1]})\frac{2C_j}{B_{jj}^2 4A_jC_j} signB$$

here

$$A_j = g_if_j - g_i(1 + g_i)^2 f_{j-1} + g_if_{j-2}$$

$$B_j = (2g_i + 1)f_j^2 - (1 + g_i)^2 f_{j-1} + g_if_{j-2}$$

$$C_j = (g_i + 1)f_j;\ f_j = f(Z^{[j]});$$

$$g_j = \frac{Z^{[j]} - Z^{[j-1]}}{Z^{[j-1]} - Z^{[j-2]}}; j = 0,1,2$$

To start solving

$$Z^{[0]} = Z00;\ Z^{[1]} = Z01;\ Z^{[2]} = Z02;$$

$Z00, Z01, Z02$ –solution of elastic problem.

When studying a surface wave (Relay wave) in a half-plane in a classical medium, it does not depend on (2). That is, the wave does not undergo dispersion. This equation (1) is a transcendental equation with a countable number of complex roots, but infinitely many imaginary roots. This equation is solved based on the algorithm and program developed in the second chapter. Obtained in calculations

$$A = 0{,}048;\ \beta = 0{,}05;\ \alpha = 0{,}10$$

Currently, an effective way to take into account the viscosity property is the three-parameter Rjanitsyn-Koltunov relaxation kernel, which takes the Poisson's ratio as constant.

$$R_j(t) = A_je^{-\beta_jt}/t^{1-\alpha_j}$$

Influence function

$$R_j(t - \tau)$$

The usual conditions for integrality, continuity ($t = \tau$ except for a point), definite sign, and monotonicity are as follows:

$$R > 0, \frac{dR(t)}{dt} \le 0, 0 < \int_0^\infty R(t)dt < 1$$

The following quantities were used to obtain numerical results

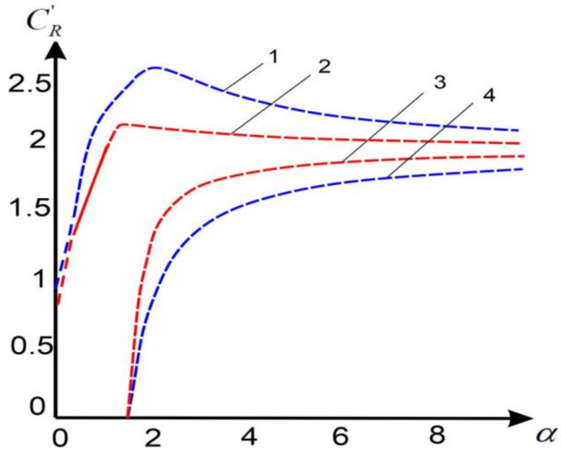

Figure 1. Change in the actual part of the surface wave velocity depending on the wave number:1,4-surface wave velocity; 2.3 - phase velocity of the displacement wave

The change in the real and imaginary parts of the surface wave velocity depending on the wave number is shown in Figure 1.

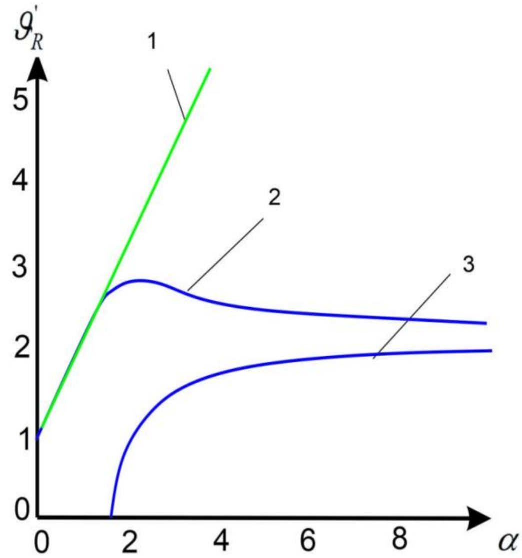

Figure 2. Change in the real and imaginary parts of the surface wave velocity depending on the wave number:1,2 - phase velocity of the displacement wave;3-surface wave velocity;

The results were obtained for dimensionless parameters. The results were obtained for two values of the Poisson's ratio. First$\nu = 0{,}25$ (continuous lines) and $\nu = 0{,}45$ (continuous lines). As can be seen from the picture, it will $\alpha \to \infty, \zeta \to 2, c_R \to \sqrt{2}$ happen.

Figure 2 below shows the change in the phase velocity c_R and the displacement or transverse) wave depending on (the wave number of the surface wave V_f.

Used in calculations ν=0,45. From the obtained results, it follows that the velocity of the surface wave does not exceed the velocity of the displacement wave. The distribution of stresses and displacements can be calculated using displacement potentials. They have the following form.

Conclusions

1. A methodology and algorithm for complex arithmetic have been developed using methods for calculating the Müller and matrix determinants. Problems were also posed through displacements (Lame's equation).

2. The dispersion relation was studied $\Omega(\omega = \omega_R + i\omega_I)$ depending on the dimensionless wavelength of the complex frequency. The absolute error $18{*}10^{-11}$ of calculating eigenvalues (the result after substituting the eigenvalue into the equation) is equal to.

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